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1992 J. Phys. A: Math. Gen. 25 5151

(http://iopscience.iop.org/0305-4470/25/19/023)

J. Phys. A: Math. Gen. 25 (1992) 5151–5163. Printed in the UK

Aharonov–Anandan geometric phase for spin-$\frac{1}{2}$ periodic Hamiltonians

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Received 29 January 1992

Abstract. We calculate exactly the Aharonov–Anandan phase for the evolution of a spin-$\frac{1}{2}$ in some periodic time-dependent magnetic fields, and give a discussion of the results versus the adiabatic ones. In particular, we analyse the existence of cyclic states and show explicit examples of systems with a prescribed value of the AA phase for all cyclic evolutions.

1. Introduction

Berry's phase [1] and its generalization, the Aharonov–Anandan (AA) phase [2], have focused the attention of many physicists in recent years [3–5]. In this paper we study the geometrical phases for the spin evolution in presence of periodic time-dependent homogeneous magnetic fields. Floquet's theory gives a good method to obtain the geometric phase after a cyclic evolution, as Moore showed recently [6, 7]. For Berry's adiabatic phase, the closed circuit is given by a cyclic change of the external Hamiltonian (in some parameter space), and the relevant quantity is the anholonomy of the circuit relative to the Berry connection. When, for instance, we consider a rotating magnetic field, the parameter space is a sphere and the Berry connection is the ordinary Levi–Civita connection for the standard metric on the sphere. This leads to the equality of the Berry phase and the area enclosed by the path of the magnetic field in the parameter space. It is interesting to remark that previous authors implicitly introduced in the formulation of the adiabatic theorem a general phase (see for instance [8]) called by some authors [9] the Born–Fock–Schiff (BFS) phase; for a periodic Hamiltonian it reduces to Berry's adiabatic phase.

For the AA phase, the structure involved is a Hilbert bundle with base the projective state space and fibre U(1). The connection in the space of states—the projective Hilbert space—is a canonical one, independent of the Hamiltonian. The AA phase appears as the holonomy in the projective space, as in general the horizontal lifting of a closed curve in the projective space is open on the fibre bundle [10]. Furthermore, in the spin-$\frac{1}{2}$ case each spin state can be represented as a point in a unit 2-sphere S^2 and its evolution is seen as a path on this surface. A cyclic evolution of the state is a closed curve on the spherical surface and the AA geometrical phase is directly

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related with the area enclosed by the curve [11] because in this case the canonical connection turns out to be the Levi–Civita connection for the standard Riemannian structure of the sphere.

A further difference is that adiabatic phases refer to the evolution of particular states under a controlled motion of external parameters, while the AA phase refers to the exact evolution of any cyclic state in a (possibly time-dependent) given Hamiltonian. In the adiabatic limit, starting from the AA phase we recover Berry's phase as a first-order term, while the following terms describe deviations from adiabaticity.

The paper is organized as follows. Section 2 gives a brief overview of the relationship between the spin state space and the sphere S^2 with the standard 2-form area. This sphere is also the phase space in the Moyal formalism of quantum mechanics [12, 13]. The quantum evolution is described by a curve in the sphere; an explicit expression for the oriented area enclosed by any closed curve is given. In section 3 we present a *general expression of the Aharonov–Anandan* phase for any cyclic evolution in a homogeneous but time-dependent magnetic field in terms of the path on the spin sphere. Some comments on Berry and BFS phases for adiabatic closed evolutions are also given. In section 4, we give completely explicit expressions for the AA phase in some specific examples of physical interest: a linear harmonic oscillating and a rotating magnetic field, both with a constant term. It is surprising that for particular magnetic fields, all states are cyclic and the geometric phase can be adjusted by adequately fixing the relative strengths and frequency of the components of the magnetic field. In particular we give conditions on the magnetic field which lead to a zero AA phase in all states, or to some prescribed values in a specific initial state. The relationship between adiabatic (Berry's and BFS phases) and the AA phase is explicitly discussed in these examples; we evaluate the adiabatic geometric phases and compare these results with the ones obtained using the Anandan and Stodolsky method [14] to compute the Berry phase.

2. Spin-$\frac{1}{2}$ systems in time-dependent magnetic fields

Let us consider a spin-$\frac{1}{2}$ charged particle in an arbitrary homogeneous magnetic field $\boldsymbol{B}(t)$. The spin term of the Hamiltonian associated to the system is

$$H(t) = -\mu \boldsymbol{B}(t) \cdot \boldsymbol{S} = -\frac{\mu\hbar}{2} \boldsymbol{B}(t) \cdot \boldsymbol{\sigma} . \tag{2.1}$$

The space of states of this system is the projective space $\mathbb{C}P^1$, which is diffeomorphic to the unit sphere S^2 [15]. The point in S^2 associated with an arbitrary state $|\psi\rangle$ of this system is $\boldsymbol{n} = \langle\psi|\boldsymbol{\sigma}|\psi\rangle$. Reciprocally, to a given vector $\boldsymbol{n} \in S^2$, parametrized in a North chart by

$$n_1 = \sin\theta\cos\phi \qquad n_2 = \sin\theta\sin\phi \qquad n_3 = \cos\theta \tag{2.2}$$

we can associate the spin state

$$|\psi\rangle = \begin{pmatrix} e^{-i\phi/2}\cos(\theta/2) \\ e^{i\phi/2}\sin(\theta/2) \end{pmatrix} . \tag{2.3}$$

The Schrödinger equation for the state $|\psi(t)\rangle$

$$\frac{d}{dt}|\psi(t)\rangle = -\frac{i}{\hbar} H(t)|\psi(t)\rangle \tag{2.4}$$

is expressed in the following form for the vector $\boldsymbol{n}(t)$:

$$\frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}t} = -\mu \boldsymbol{B}(t) \times \boldsymbol{n} . \tag{2.5}$$

Notice that the same equation can be obtained from a phase-space quantum approach to the problem [12, 13].

In general, the evolution for an initial state $\boldsymbol{n}(0)$ is a curve on the sphere, $\boldsymbol{n}(t)$, that, depending on the external field, can be very complicated. We will study some cases for which $\boldsymbol{n}(t)$ describes a closed curve, that is, for some $\tau > 0$, $\boldsymbol{n}(\tau) = \boldsymbol{n}(0)$. When this is the case, it makes sense to evaluate the area (which equals the solid angle from the point of view of the sphere imbedded in a three-dimensional ambient space) enclosed by the trajectory:

$$\Delta\Omega = \int_{\mathrm{Surface}} \boldsymbol{n} \cdot \mathrm{d}\boldsymbol{S} .$$

An easy way to compute that magnitude is to transform the surface integral into a line integral. Therefore, we need a vector field $\boldsymbol{A}(\boldsymbol{r})$ such that $\nabla \times \boldsymbol{A} = \boldsymbol{n}$ on the surface. It is well known that the vector potential describing the monopole field satisfies this requirement. In terms of cartesian coordinates $\boldsymbol{r} = (x_1, x_2, x_3)$, this vector potential is given by

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{1}{r}\left(\frac{-x_2}{r + x_3}, \frac{x_1}{1 + x_3}, 0\right) \qquad \nabla \times \boldsymbol{A} = \frac{\boldsymbol{r}}{r^3} . \tag{2.6}$$

Making use of the Stokes theorem, the total solid angle subtended by the oriented closed curve $\boldsymbol{n}(t)$ is

$$\Delta\Omega = \int_{\mathrm{Surface}} \boldsymbol{n} \cdot \mathrm{d}\boldsymbol{S} = \oint_C \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{n} = \int_0^\tau \frac{n_1 \dot{n}_2 - n_2 \dot{n}_1}{1 + n_3}\,\mathrm{d}t \tag{2.7}$$

where n_1, n_2, n_3 are the components of $\boldsymbol{n}$ and $\dot{n}_1$, $\dot{n}_2$ denote time derivatives.

3. The geometric phases

Let us first discuss the AA phase. Take a generic initial state $|\psi(0)\rangle$ given by

$$|\psi(0)\rangle = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\phi_0/2}\cos(\theta_0/2) \\ \mathrm{e}^{\mathrm{i}\phi_0/2}\sin(\theta_0/2) \end{pmatrix} . \tag{3.1}$$

At the instant t the state $|\psi(t)\rangle$ will be

$$|\psi(t)\rangle = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\phi(t)/2}\cos(\theta(t)/2) \\ \mathrm{e}^{\mathrm{i}\phi(t)/2}\sin(\theta(t)/2) \end{pmatrix} \tag{3.2}$$

with $\theta(0) = \theta_0$ and $\phi(0) = \phi_0$. If we now assume that the evolution of the initial state $|\psi(0)\rangle$ is cyclic with period τ, then $\theta(t)$ must be a τ-periodic function and we should have $|\psi(\tau)\rangle = \mathrm{e}^{\mathrm{i}\alpha(\tau)}|\psi(0)\rangle$. Comparing (3.1) and (3.2), it is straightforward to prove that $\exp\{\mathrm{i}\alpha(\tau)\} = \exp\{\mathrm{i}[\phi(\tau) - \phi(0)]/2\} = \exp\{-\mathrm{i}[\phi(\tau) - \phi(0)]/2\}$, so that $\phi(\tau) - \phi(0) = 2\pi m$, for some $m \in \mathbb{Z}$. The AA phase β associated to the evolution between $t = 0$ and $t = \tau$ is given by the well known formula [2]

$$\beta = -\frac{\phi(\tau) - \phi(0)}{2} + \mathrm{i} \int_0^\tau \left\langle \psi(t) \left| \frac{\mathrm{d}}{\mathrm{d}t} \right| \psi(t) \right\rangle \mathrm{d}t = -\frac{1}{2} \int_0^\tau [\dot{\phi}(t) - 2\mathrm{i}\langle \psi(t) | \dot{\psi}(t) \rangle] \, \mathrm{d}t \,. \tag{3.3}$$

A simple computation leads to

$$\beta = -\frac{1}{2} \int_0^\tau \dot{\phi}(1 - \cos\theta) \, \mathrm{d}t \,. \tag{3.4}$$

Introducing now cartesian coordinates (n_1, n_2, n_3) in the ambient space, it is clear that $\phi = \tan^{-1}(n_2/n_1)$ and taking into account that $\boldsymbol{n}^2 = 1$,

$$\dot{\phi} = \frac{n_1 \dot{n}_2 - n_2 \dot{n}_1}{1 - n_3^2} \,. \tag{3.5}$$

Finally, substituting (2.2) and (3.5) in (3.4), we obtain the following expression for β:

$$\beta = -\frac{1}{2} \int_0^\tau \frac{n_1 \dot{n}_2 - n_2 \dot{n}_1}{1 + n_3} \, \mathrm{d}t \,. \tag{3.6}$$

Comparing (2.7) and (3.6), we have shown that

$$\beta = -\tfrac{1}{2} \Delta\Omega \,. \tag{3.7}$$

This result was already known [11], but our derivation can be illustrative to the reader. The determination of the cyclic states obeying the Schrödinger equation (2.4) or (2.5) and the evaluation of (3.6) for their AA geometric phases depends on the explicit form for $\boldsymbol{B}(t)$ in (2.1), and, in general, is not an easy task.

Consider now a slowly variable periodic magnetic field $\boldsymbol{B}(t)$, so that the adiabatic theorem is applicable. If the initial state is a proper state of the initial Hamiltonian (i.e. the spin state points along $\boldsymbol{B}(0)$ or in the opposite direction), because $\boldsymbol{B}(t)$ changes in time adiabatically, the spin associated point will follow the field $\boldsymbol{B}(t)$ at each t, i.e.

$$\boldsymbol{n}_\pm^a(t) = \pm \boldsymbol{B}(t) / |\boldsymbol{B}(t)| \,. \tag{3.8}$$

In particular, when the field $\boldsymbol{B}(t)$ returns to its initial value, say at $t = \tau$, the two states $\boldsymbol{n}_\pm^a(t)$ will also return to their initial positions on S^2, $\boldsymbol{n}_\pm^a(\tau) = \boldsymbol{n}_\pm^a(0)$ and close a circuit in the spin space state. It is not a surprise to find that the value of the AA phase for this circuit (as given by (3.6) and (3.7)) equals the adiabatic Berry phase, thus making clear the equality of the area enclosed by the loop $\boldsymbol{n}_\pm^a(t)$ on S^2 in the spin state space with the solid angle swept by the path of the magnetic field $\pm\boldsymbol{B}(t)$ in the three-dimensional parameter space of values of the magnetic field.

In order to compare for some particular examples the phases obtained by these techniques with the ones calculated by using the adiabatic theorem, we now summarize the prescription given by Anandan and Stodolsky [14] for an adiabatically varying τ-periodic Hamiltonian of the form (2.1).

Let $|E_\pm(0)\rangle$ be the eigenstates of $H(0) = -\mu \boldsymbol{B}(0)\cdot \boldsymbol{S}$ with eigenvalues $E_\pm(0)$. These two states point respectively along the magnetic field $\boldsymbol{B}(0)$ and in the opposite direction. Suppose that the τ-periodic unitary operator $D(t)$, with $D(\tau) = D(0) = I$, diagonalizes $H(t)$ in the base of eigenstates $|E_\pm(t)\rangle = D(t)|E_\pm(0)\rangle$, i.e.

$$H(t)|E_\pm(t)\rangle = E_\pm(t)|E_\pm(t)\rangle. \tag{3.9}$$

Starting from the states $|E_\pm(t)\rangle$ we construct a new base $|\widehat{E_\pm(t)}\rangle = e^{i\gamma_\pm(t)}|E_\pm(t)\rangle$ satisfying the phase fixing conditions [8]

$$\left\langle \widehat{E_\pm(t)} \middle| \frac{d}{dt} \middle| \widehat{E_\pm(t)} \right\rangle = 0 . \tag{3.10}$$

The $\gamma_\pm(t)$, known as the Born–Fock–Schiff (BFS) phases, read:

$$\gamma_\pm(t) = i\int_0^t \left\langle E_\pm(t') \middle| \frac{d}{dt'} \middle| E_\pm(t') \right\rangle dt' = i\int_0^t \left\langle E_\pm(0) \middle| D^\dagger(t')\frac{dD(t')}{dt'} \middle| E_\pm(0) \right\rangle dt' . \tag{3.11}$$

The adiabatic evolution induced by the Schrödinger equation on $|E_\pm(0)\rangle$ is:

$$\begin{aligned} |\psi_\pm(t)\rangle_a &= \exp\left(-\frac{i}{\hbar}\int_0^t E_\pm(t')dt'\right)|\widehat{E_\pm(t)}\rangle \\ &= \exp\left(-\frac{i}{\hbar}\int_0^t E_\pm(t')dt'\right)e^{i\gamma_\pm(t)}|E_\pm(t)\rangle . \end{aligned} \tag{3.12}$$

In contrast with the base $|E_\pm(t)\rangle$ which is chosen τ-periodic, the base $|\widehat{E_\pm(t)}\rangle$ does not come back, in general, to its initial values at $t = \tau$:

$$|\widehat{E_\pm(\tau)}\rangle = e^{i\gamma_\pm(\tau)}|E_\pm(\tau)\rangle = e^{i\gamma_\pm(\tau)}|E_\pm(0)\rangle . \tag{3.13}$$

Thus, the Berry phases $\gamma_\pm(\tau)$ are absorbed in the base $|\widehat{E_\pm(t)}\rangle$ because the phase fixing requirement; in the Berry original formulation the same phase factor was needed to compensate the absence of the mentioned condition on the base $|E_\pm(t)\rangle$ (for an extensive discussion of this subject see [9]).

In the following we will calculate both AA and adiabatic phases explicitly and the general relation (3.7) will be verified for some interesting examples.

4. Examples

4.1. Linear oscillating magnetic field

Let us suppose that the spin-$\frac{1}{2}$ evolves in the presence of the following oscillating magnetic field:

$$\boldsymbol{B}(t) = (0, 0, B_0 + B\cos\omega t) . \tag{4.1}$$

The evolution equation for the Hamiltonian (2.1), with $\boldsymbol{B}(t)$ given by (4.1), reads:

$$\frac{\mathrm{d}}{\mathrm{d}t}U(t) = -\frac{\mathrm{i}}{\hbar}H(t)U(t) \qquad U(0) = I \tag{4.2}$$

$U(t)$ being the evolution operator of the system, $|\psi(t)\rangle = U(t)|\psi(0)\rangle$. The exact solution is given by the standard expression $U(t) = \exp[-\frac{\mathrm{i}}{\hbar}\int_0^t H(t')\mathrm{d}t']$ because the Hamiltonians at different times commute. An explicit calculation gives

$$U(t) = \begin{pmatrix} \exp\left[\mathrm{i}\left(\frac{b_0 t}{2} + \frac{b}{2\omega}\sin\omega t\right)\right] & 0 \\ 0 & \exp\left[-\mathrm{i}\left(\frac{b_0 t}{2} + \frac{b}{2\omega}\sin\omega t\right)\right] \end{pmatrix} \tag{4.3}$$

with $b_0 = \mu B_0$ and $b = \mu B$. Of course, the same result can be found by writing $U(t)$ as

$$U(t) = g_0(t)I + \sum_{i=1}^{3} g_i(t)\sigma_i \tag{4.4}$$

(because $I, \sigma_1, \sigma_2, \sigma_3$ are a base for the vector space of 2×2 matrices) and solving the system of ordinary differential equations for (g_0, g_1, g_2, g_3).

Choosing (3.1) as initial state (with $\phi_0 = 0$ and disregarding in future the subscript of the angle θ_0) we have

$$|\psi(t)\rangle = \begin{pmatrix} \exp\left[\mathrm{i}\left(\frac{b_0 t}{2} + \frac{b}{2\omega}\sin\omega t\right)\right]\cos\frac{\theta}{2} \\ \exp\left[-\mathrm{i}\left(\frac{b_0 t}{2} + \frac{b}{2\omega}\sin\omega t\right)\right]\sin\frac{\theta}{2} \end{pmatrix}. \tag{4.5}$$

For given b_0, b, ω, this state becomes cyclic for a set of times τ_m labelled by an integer m, satisfying the transcendental equation

$$\frac{b_0 \tau_m}{2} + \frac{b}{2\omega}\sin\omega\tau_m = m\pi \qquad m \in \mathbb{Z}. \tag{4.6}$$

We then have $|\psi(\tau_m)\rangle = \mathrm{e}^{\mathrm{i}m\pi}|\psi(0)\rangle$, and its associated geometric phase is, according to (3.3):

$$\beta_m = m\pi(1 - \cos\theta). \tag{4.7}$$

In order to evaluate the adiabatic phases we note that in the parameter space of values of the magnetic field, the closed path followed by $\boldsymbol{B}(t)$ is a segment of a straight line along the z-axis with period $\tau = 2\pi/\omega$. This path encloses no solid angle, so that the adiabatic phases $\beta_\pm^a$ are equal to zero for the two adiabatic states $\boldsymbol{n}_\pm^a(t)$ in (3.8). On the other hand, the application of the Anandan–Stodolsky prescription is trivial in this example by choosing for $|E_+(0)\rangle$ and $|E_-(0)\rangle$ the usual S_z up and down states and $D(t) = I$. The initial basis satisfies the phase fixing condition for all t, so the BFS phases are null, $\gamma_\pm(t) = 0$, and the Berry phases for the two adiabatic cyclic states are also equal to zero. These results agree with the ones obtained by evaluating the areas enclosed by the trajectories $\boldsymbol{n}_\pm^a(t)$ in S^2 or by

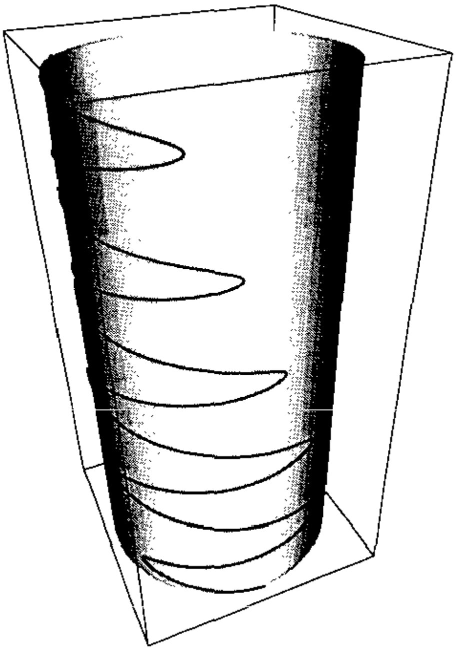

Figure 1. The projection on the x–y plane of the spin trajectory $\boldsymbol{n}(t)$ versus time (vertical axis) in the oscillating magnetic field (4.1). The parameters of the system were taken as $b/\omega = 1$ and $b_0/\omega = \frac{1}{20}$ and the radius of the cylinder is $\rho = \sin\theta$. In the sphere of states S^2, $\boldsymbol{n}(t)$ moves on a cone with vertex at the origin and cone semiangle θ, returning to its initial condition for all $t = \tau_m$ where the curve $\boldsymbol{n}(t)$ intersects the vertical line $(\sin\theta, 0, t)$. For these times, the AA phase involves just the effective number of rotations (m) that the spin has performed.

the cyclic paths $\pm\boldsymbol{B}(t)$ in the parameter space. All other states are not cyclic in the adiabatic approximation.

However, the exact solution of the system is somewhat different: for some specific values of the time interval *all states* are cyclic, and the path $\boldsymbol{n}(t)$ followed by the state is

$$
\begin{aligned}
n_1(t) &= \sin\theta\cos\left(b_0 t + \frac{b}{\omega}\sin\omega t\right) \\
n_2(t) &= -\sin\theta\sin\left(b_0 t + \frac{b}{\omega}\sin\omega t\right) \\
n_3(t) &= \cos\theta\,.
\end{aligned}
\tag{4.8}
$$

We see that $\boldsymbol{n}$ moves on a cone with vertex at the origin and cone semiangle θ (see figure 1). For $t = \tau_m$ satisfying the condition (4.6), $\boldsymbol{n}(t)$ reaches again its initial value $\boldsymbol{n}(0)$ after m complete turns on the cone. The natural geometric quantity on S^2 associated to this evolution (no matter how it has been performed) is the area (2.7) enclosed by the closed trajectory, $\Delta\Omega = -2m\pi(1 - \cos\theta)$, which coincides with m times the solid angle subtended by the cone. This result and (4.7) verify the general relation (3.7), as expected. In particular, the exact evolution of the two adiabatic states $\theta_0 = 0, \pi$, provides an AA phase as given by (4.7)

$$\beta_m = \begin{cases} 0 & \text{for } \theta_0 = 0 \\ 2m\pi & \text{for } \theta_0 = \pi \,. \end{cases}$$

The phase factor $e^{i\beta_m}$ is always equal to 1, and therefore coincides with the Berry phase factor.

It is interesting to observe in this example the existence, for arbitrary initial conditions, of cyclic evolutions with a null geometric phase. For fixed b_0, b, ω, let τ_0 be the solution of (4.6) with $m = 0$, and let γ the solution of the transcendental equation $\tan\gamma = \pi + \gamma$, $0 < \gamma < \pi/2$. In that equation, the linear term dominates the oscillating one in the interval $-1 \leqslant (b/b_0) < 1/\cos\gamma \approx 4.60334$ and, therefore, there is no solution for $\tau_0 \neq 0$. However, outside this interval we can always find at least a value of τ_0 for which the state returns to the initial condition but without geometric phase.

4.2. *Magnetic field rotating on a cone*

In the previous problem, quite general states are cyclic but, in other situations, the set of cyclic states is rather narrow. Therefore, the determination of these states is very important. As an example, let us consider again the spin-$\frac{1}{2}$, but now with an external magnetic field precessing around the z-axis with a constant angular velocity ω:

$$\boldsymbol{B} = (B\cos\omega t,\ B\sin\omega t,\ B_0). \tag{4.9}$$

In this standard case, the parameter space is the sphere S^2 of all magnetic fields of constant module and the path in this space is a circle around the z-axis. The eigenstates of the initial Hamiltonian are those of $b\sigma_1 + b_0\sigma_3$ (where we denote again $b = \mu B$ and $b_0 = \mu B_0$). The Berry phase for these adiabatic states is very well known, and equals to the solid angle $2\pi(1 - \cos\xi_\pm)$ enclosed by the cone of semiangle $\xi_\pm$ ($\tan\xi_+ = b/b_0$, $\xi_- = \pi - \xi_+$). Other states are not cyclic in the adiabatic approximation.

We will solve the evolution equation (4.2) with $H(t)$ in the form (2.1) and the given $\boldsymbol{B}$. Note that this Hamiltonian can be expressed as

$$H(t) = -\frac{\hbar}{2}[b(\cos\omega t\sigma_1 + \sin\omega t\sigma_2) + b_0\sigma_3] = -\frac{\hbar}{2}e^{-i\omega t\sigma_3/2}(b\sigma_1 + b_0\sigma_3)e^{i\omega t\sigma_3/2}\,. \tag{4.10}$$

Here, the Hamiltonians at different times do not commute, and we have to resource to a different method to solve (4.2). If we take $U(t) = e^{-i\omega t\sigma_3/2}W(t)$ (this transformation physically represents a 'transition to a rotating frame' [16–18]), then $W(t)$ obeys a time–independent evolution equation

$$\frac{d}{dt}W(t) = \frac{i}{2}[b\sigma_1 + (b_0 + \omega)\sigma_3]W(t) \qquad W(0) = I\,. \tag{4.11}$$

Solving (4.11) we obtain

$$\begin{aligned} U(t) &= e^{-i\omega\sigma_3 t/2}\, e^{i(b\sigma_1 + b_1\sigma_3)t/2} \\ &= \begin{pmatrix} \left(\cos\frac{\omega_c t}{2} + \frac{ib_1}{\omega_c}\sin\frac{\omega_c t}{2}\right)e^{-i\omega t/2} & \frac{ib}{\omega_c}\sin\frac{\omega_c t}{2}e^{-i\omega t/2} \\ \frac{ib}{\omega_c}\sin\frac{\omega_c t}{2}e^{i\omega t/2} & \left(\cos\frac{\omega_c t}{2} - \frac{ib_1}{\omega_c}\sin\frac{\omega_c t}{2}\right)e^{i\omega t/2} \end{pmatrix} \end{aligned} \tag{4.12}$$

with $b_1 = b_0 + \omega$ and $\omega_c = \sqrt{b^2 + b_1^2}$. Choosing (3.1) as the initial condition, we cannot adjust t to achieve a cyclic evolution of $|\psi(t)\rangle$ for any θ. However, there exist two special initial states whose evolution is cyclic. At first sight, one could expect these being the (adiabatic) eigenstates of $b\sigma_1 + b_0\sigma_3$, because the spin is aligned with the magnetic field at the initial time. But this is not so, and the cyclic states are instead the eigenstates of $b\sigma_1 + b_1\sigma_3$ corresponding to the eigenvalues $\lambda_\pm = \pm\omega_c$. We shall denote them as $|\psi_\pm\rangle$ and their angles $\chi_\pm$ are given by $\chi_+ = \tan^{-1}(b/b_1)$ and $\chi_- = \pi - \chi_+$. Their evolution is quite simple:

$$|\psi_\pm(t)\rangle = \mathrm{e}^{\mathrm{i}\lambda_\pm t/2}\begin{pmatrix} \mathrm{e}^{-\mathrm{i}\omega t/2}\cos\frac{\chi_\pm}{2} \\ \pm\mathrm{e}^{\mathrm{i}\omega t/2}\sin\frac{\chi_\pm}{2} \end{pmatrix}. \tag{4.13}$$

For $\tau = 2\pi/\omega$ the states are cyclic:

$$|\psi_\pm(\tau)\rangle = \mathrm{e}^{-\mathrm{i}\pi(1-\lambda_\pm/\omega)}|\psi_\pm\rangle = \mathrm{e}^{\mathrm{i}\phi_\pm}|\psi_\pm\rangle \tag{4.14}$$

and their geometric phase reads

$$\beta_\pm = -\pi(1 - \langle\psi_\pm|\sigma_3|\psi_\pm\rangle) = -\pi(1 - \cos\chi_\pm). \tag{4.15}$$

In the adiabatic limit, the cyclic states become those pointing in the instantaneous direction of $\boldsymbol{B}(t)$ (see (3.8)). A direct calculation of the solid angles generated by the oriented curves $\boldsymbol{n}^a_\pm(t)$ on S^2 leads to the usual expressions $\beta^a_\pm = -\pi(1 - \cos\xi_\pm)$. This same result can be reached by using the Anandan–Stodolsky method. The eigenstates of the Hamiltonian at $t = 0$ and the matrix $D(t)$ which diagonalizes $H(t)$ in this basis are given by

$$|E_\pm(0)\rangle = \begin{pmatrix} \cos\xi_\pm/2 \\ \pm\sin\xi_\pm/2 \end{pmatrix} \qquad D(t) = \mathrm{e}^{\mathrm{i}\omega t/2}I\mathrm{e}^{-\mathrm{i}\omega t\sigma_3/2} = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\omega t} \end{pmatrix}. \tag{4.16}$$

The τ-periodic base (used in the Berry approach to the adiabatic approximation) reads

$$|E_\pm(t)\rangle = \begin{pmatrix} \cos\xi_\pm/2 \\ \pm\mathrm{e}^{\mathrm{i}\omega t}\sin\xi_\pm/2 \end{pmatrix}. \tag{4.17}$$

A direct calculation (see (3.11)) leads to the following BFS phases which make the base $|\widehat{E_\pm(t)}\rangle$ satisfy the phase fixing condition:

$$\gamma_\pm(t) = -\frac{\omega t}{2}(1 - \cos\xi_\pm) = -\frac{\omega t}{2}(1 \mp \cos\xi_+). \tag{4.18}$$

For $t = \tau = 2\pi/\omega$ this relation gives the adiabatic Berry phases $\gamma_\pm(\tau) = -\pi(1 - \cos\xi_\pm)$ which are the same as that obtained by the application of our general formulae to the two adiabatic eigenstates $\boldsymbol{n}^a_\pm(t)$ of the system.

The geometric character of $\beta_\pm$ is clear, working directly with (2.5), which in this case takes the form

$$\dot{\boldsymbol{n}}(t) = \begin{pmatrix} 0 & b_0 & -b\sin\omega t \\ -b_0 & 0 & b\cos\omega t \\ b\sin\omega t & -b\cos\omega t & 0 \end{pmatrix}\boldsymbol{n}(t). \tag{4.19}$$

To solve this system, let us consider the new vector

$$a(t) = \begin{pmatrix} \cos\omega t & \sin\omega t & 0 \\ -\sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} n(t)\,. \tag{4.20}$$

The differential equation for $a(t)$ is time-independent:

$$\dot{a}(t) = \begin{pmatrix} 0 & b_1 & 0 \\ -b_1 & 0 & b \\ 0 & -b & 0 \end{pmatrix} a(t) \qquad a(0) = n(0)\,. \tag{4.21}$$

The solution of (4.21) provides n:

$$n(t) = \begin{pmatrix} \cos\omega t & -\sin\omega t & 0 \\ \sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$\times \begin{pmatrix} \left(\frac{b}{\omega_c}\right)^2 + \left(\frac{b_1}{\omega_c}\right)^2 \cos\omega_c t & \frac{b_1}{\omega_c}\sin\omega_c t & \frac{bb_1}{\omega_c^2}(1-\cos\omega_c t) \\ -\frac{b_1}{\omega_c}\sin\omega_c t & \cos\omega_c t & \frac{b}{\omega_c}\sin\omega_c t \\ \frac{bb_1}{\omega_c^2}(1-\cos\omega_c t) & -\frac{b}{\omega_c}\sin\omega_c t & \left(\frac{b_1}{\omega_c}\right)^2 + \left(\frac{b}{\omega_c}\right)^2\cos\omega_c t \end{pmatrix} n(0)\,. \tag{4.22}$$

Note that the evolution given by (4.22) is basically the superposition of two rotations. The first one takes place around an axis placed in the x–z plane, at an angle χ_+ from the z-axis and having angular velocity ω_c. The second one is a precession around the z-axis with angular velocity ω. Due to the fact that the frequencies ω_c and ω are, in general, non-commensurable, the motion for an arbitrary initial state $n(0)$ is not cyclic. The only exceptions are the trajectories in which $n(0)$ is aligned in the direction defined by χ_+:

$$n_\pm(0) = \pm \begin{pmatrix} \sin\chi_+ \\ 0 \\ \cos\chi_+ \end{pmatrix}\,. \tag{4.23}$$

These two vectors are the representatives on the sphere S^2 of the two states $|\psi_\pm(0)\rangle$. The vectors $n_\pm(t)$ move around the z-axis with an angular velocity ω, and for $\tau = 2\pi/\omega$ they return to their initial values, generating a solid angle (equation (2.7)) $\Delta\Omega_\pm = 2\pi(1-\cos\chi_\pm) = -2\beta_\pm$.

Let us remark once again that these phases are different from the solid angle enclosed by the trajectory of the magnetic field in the parameter space $[2\pi(1-\cos\xi_\pm)]$. The exact evolution of the proper states of the instantaneous Hamiltonian is not cyclic, in general. But, of course, in the adiabatic limit ($\omega \to 0$) our cyclic states go over the proper states of the instantaneous Hamiltonian because $\chi_\pm \to \xi_\pm$ in this limit.

It is interesting to obtain the BFS phases and states for the exact evolution (not necessarily adiabatic). We choose as initial conditions the eigenstates $|\psi_\pm\rangle$ of $b\sigma_1 + b_1\sigma_3$ because their corresponding evolution is cyclic. Note that the adiabatic limit of the AA phases and eigenvectors of $b\sigma_1 + b_1\sigma_3$ go over the adiabatic phases and eigenbasis of $H(0)$; then we can use $|\psi_\pm(t)\rangle$ to construct the generalization. We fix as the τ-periodic base (which is proportional to $|\psi_\pm(t)\rangle$ at each t)

$$|\mathcal{E}_\pm(t)\rangle = \begin{pmatrix} \cos\chi_\pm/2 \\ \pm e^{i\omega t}\sin\chi_\pm/2 \end{pmatrix} . \tag{4.24}$$

Now we construct the base $|\widehat{\mathcal{E}_\pm(t)}\rangle = e^{i\beta_\pm(t)}|\mathcal{E}_\pm(t)\rangle$ which satisfies the phase fixing requirement $\langle\widehat{\mathcal{E}_\pm(t)}|\frac{d}{dt}|\widehat{\mathcal{E}_\pm(t)}\rangle = 0$. Henceforth

$$\beta_\pm(t) = i\int_0^t \left\langle \mathcal{E}_\pm(t')\left|\frac{d}{dt'}\right|\mathcal{E}_\pm(t')\right\rangle dt' = -\frac{\omega t}{2}\left(1-\cos\chi_\pm\right) . \tag{4.25}$$

The generalized BFS phases $\beta_\pm(t)$ provide the AA phases $\beta_\pm(\tau) = -\pi\left(1-\cos\chi_\pm\right)$ as we can see comparing this result with (4.15).

If $\chi_\pm$ is considered as a function of ω, from (4.15) we get an exact expression for the AA phases in powers of the adiabatic parameter ω. The first terms of this expression are

$$\beta_\pm = -\pi(1-\cos\xi_\pm) \pm \frac{\pi b^2}{(b^2+b_0^2)^{3/2}}\omega \mp \frac{3\pi b^2 b_0}{2(b^2+b_0^2)^{5/2}}\omega^2 \pm \cdots \tag{4.26}$$

where, of course, the first term is the Berry phase, but the next are 'non-adiabatic' corrections to that phase.

An interesting question can be posed: what happens when ω_c and ω in the previous situation are commensurable? If this is the case, we can write $\omega_c = \omega l/n$, where $l,\ n \in \mathbb{N}$. Any initial state in the form (3.1) performs a cyclic evolution according to (4.12), with period $\tau = 2n\pi/\omega$, $|\psi(\tau)\rangle = e^{i(l-n)\pi}|\psi(0)\rangle$. The associated geometric phase can be evaluated explicitly:

$$\beta = l\pi[1-\cos(\theta-\chi_+)] - n\pi[1-\cos\chi_+\cos(\theta-\chi_+)] . \tag{4.27}$$

So, in this case *all* states have a cyclic evolution, with a period which is some multiple of the period of the external magnetic field; this happens only for a specific relation between the relative amplitudes of the constant and rotating parts of the magnetic field.

In S^2, the vector $\boldsymbol{n}(t)$ is again cyclic, and does not now sweep a cone, as in the two cases previously discussed (see figure 2), but the spherical analogue of an epicycloid or an hypocycloid. However, the area enclosed can be explicitly evaluated from (2.7) and (4.22):

$$\Delta\Omega = 2n\pi[1-\cos\chi_+\cos(\theta-\chi_+)] - 2l\pi[1-\cos(\theta-\chi_+)] = -2\beta . \tag{4.28}$$

For given values of n, l, there may exist cyclic states whose geometric phase vanishes. This occurs when θ in (3.1) takes the value

$$\theta = \chi_+ + \cos^{-1}\left(\frac{n-l}{n\cos\chi_+ - l}\right) \qquad 0 \leqslant \theta \leqslant \pi . \tag{4.29}$$

If $l \geqslant n$, this equation has a solution for any value of χ_+. For $l < n$, that equation admits solution only in the case that χ_+ verifies $2l \geqslant n(1+\cos\chi_+)$. We can always select the physical parameters so that this inequality is accomplished.

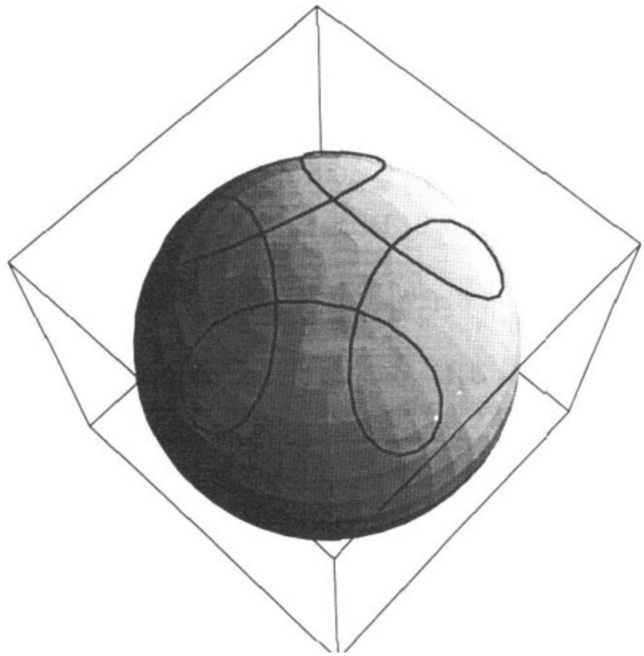

Figure 2. A closed trajectory for the spin in the precessing magnetic field (4.9) in the case in which the frequencies ω_c and ω are commensurable. The chosen parameters are $b/\omega = 3$ and $b_0/\omega = 3$ so that $b_1/\omega = 4$ and $\omega_c/\omega = 5$, while the initial conditions are $n(0) = (\frac{\sqrt{3}}{2}, 0, \frac{1}{2})$. The global period is $\tau = 2\pi/\omega$ and in the rotating frame the system performs five rotations around an axis on the x–z plane at the angle $\chi_+ = \tan^{-1}(\frac{3}{4})$ off the z-axis. The evolution in the original frame is obtained by combining this with a rotation around the z-axis, thereby obtaining an spherical epicycloid.

5. Concluding remarks

We have explicitly studied AA and adiabatic (BFS and Berry) phases for a spin-$\frac{1}{2}$ in some magnetic fields. When the AA phase is developed in powers of the adiabatic parameter, then the first order in the development corresponds to the Berry phase, and the other terms can be considered as 'non-adiabatic' corrections to that phase. Thus in the adiabatic evolution both phases agree; however some care must be exercised in the comparison as in general the adiabatic eigenstates of the Hamiltonian are not cyclic. It is also interesting to remark that for particular values of the magnetic field parameters all the states are cyclic, and the geometric phases can be brought to prescribed values, including 0.

Here we have considered only the spin part of the Hamiltonian. Another interesting system is a spinless particle in a (periodic) electromagnetic field. Now there is no spin–magnetic field coupling, but the Hamiltonian gets a term from the minimal electromagnetic substitution $p - (e/c)A(r, t)$. In a forthcoming paper we will study this case, where a geometric phase will also appear for 'cyclic' evolutions.

Acknowledgments

We acknowledge support from CICYT (Spain) under reference AEN90-0027 and from Caja Salamanca. One of the authors (DJFC) also acknowledges financial support from the Instituto de Cooperación Iberoamericana of the Agencia Española de Cooperación Internacional (Spain).

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